Modeling of corneal deformation under air-puff by nonlinear differential equations

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Abstract

Intraocular Pressure (IOP) is a main factor for the diagnosis of glaucoma. In this report, the kinematic viscoelastic corneal models, specifically the Maxwell and the Kelvin-Voight models, of human eye ball will be proposed for determining the displacement of the cornea during the air-puff tonometry simulations and its relationship to IOP. The purpose of project is to study the influence of elasticity and viscosity to the corneal deformations under an air puff.

Introduction

The eyes are one of the most complex and important organs that provide us with vision. Eyes detect the light and converse it into electrochemical signals. The primary function of the cornea is to focus the light on the retina for producing an accurate image. Glaucoma is a condition in which the optic nerve, which is responsible for the transmission of images to the brain, is damaged, and without treatment may lead to a permanent blindness [1]. The high intraocular pressure (IOP) is a main cause of glaucomatous damage to the optic nerves in the lamina cribrosa (LC) (Fig. 1).
In this report we intend to use some simple mechanical and mathematical models to analyze and assess the corneal movements and the stress-strain dependence of corneal tissue.

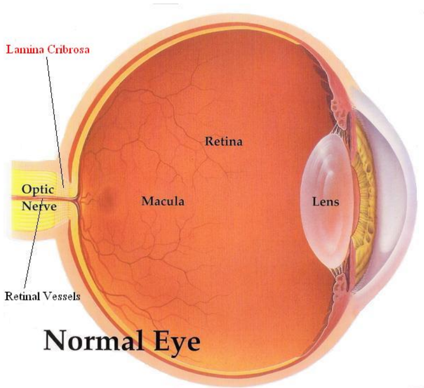


Figure 1: The ocular structure [2].

Modeling one dimensional human eye ball

In recent years, researchers have began to use computational models to understand the biomechanical properties of the eye and the linkage between IOP and glaucoma [3]. The cornea of the eye has a nonlinear viscoelastic behaviour and requires nonlinear structural models that captures stiffening induced by the relation of stress and strain. The cornea allows for an immediate deformation when the external force applied and then followed by a progressive deformation. This process can be simply described with a simple spring and dashpot system.

The Kelvin-Voight model

For cases when time-dependent effects cannot be neglected a viscoelasticity constitutive models should be utilized. Kelvin-Voight (also known as Voight model) is a viscoelastic model that has both elasticity and viscosity properties and it is represented by a viscous damper and elastic spring in parallel as shown in Fig. 2(b). The strains of parallel components are identical. When the stress is released, elastic component changes gradually to its undeformed state and does not allow for an immediate deformation.

Given the nonlinear dynamic model [4]:

$$\begin{cases} m\frac{d^2y(t)}{dt^2} + c\frac{dy(t)}{dt} + ky(t) = F(t) = F_{IOP} - F_{air} \\ y(0) = F_{IOP}/k \\ y(T) = F_{IOP}/k \end{cases} \tag{1}$$

where A, the effective area for the IOP with radius, r, equals to 1.5 mm and $0 \leq t \leq 0.03$.

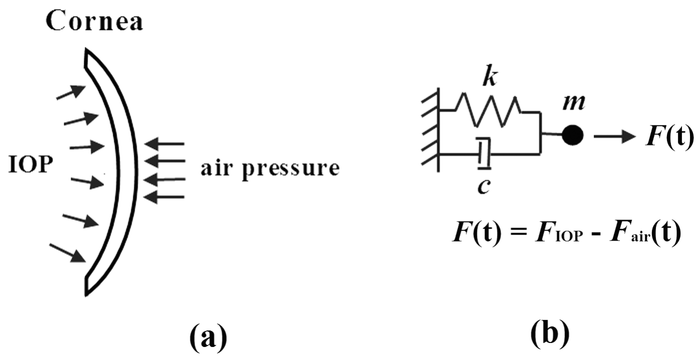


Figure 2: (a) An illustration of the cornea under intraocular pressure (IOP) and air pressure. (b) A kinematic model of the cornea under air puff deformation [4].

F(t) is a total external force applied on the cornea and it is divided into two parts (see Eq. (2)): the one comes from the intraocular pressure, F_{IOP}, and from the air puff force, $F_{air}(t)$:

$$F(t) = F_{IOP} - F_{air}(t) \tag{2}$$

where $F_{IOP} = IOP * A$ and $F_{air}(t) = P_{air}(t) * A_{air}$. The measurement of IOP made by Goldmann applanation tonometer equals to 13.8 mmHg and the effective area for the IOP.

The spatial radius of the air puff A_{air} is 1.5 mm and $A_{air} = \pi r^2$. $P_{air}(t)$ is the air pressure that changes with time and have the following function:

$$P_{air}(t) = a_1 e^{-(\frac{t-b_1}{c_1})^2} + a_2 e^{-(\frac{t-b_2}{c_2})^2} \tag{3}$$

where a_1, a_2, b_1, b_2, c_1, c_2 are parameters obtained from [4] and their corresponding values are $a_1 = 5149$, $b_1 = 0.01752$, $c_1 = 0.00441$, $a_2 = 3398$, $b_2 = 0.01167$ and $c_2 = 0.04311$. For the graphical representation of Eq.(2) refer to Fig.2. The corneal elasticity coefficient, k, is 85 N/m [5]. The viscoelasticity coefficient was set between 0.01 and 0.26 Ns/m [5].

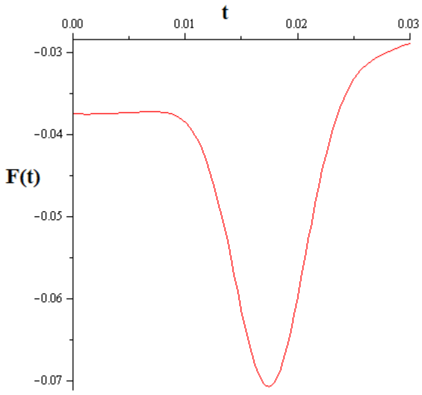
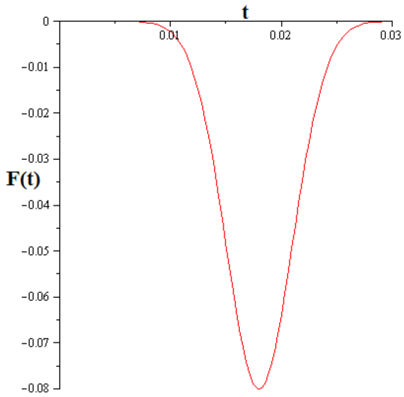


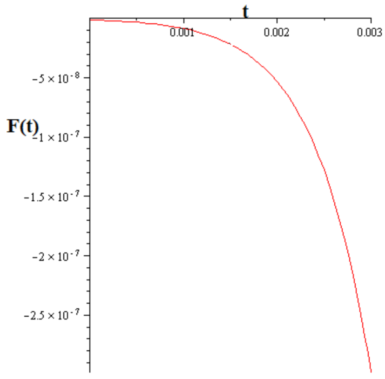
Figure 3: Total external force of IOP and air puff.

The second external force (also called as force model 2) is based on the experimental results that represents the bell shaped curve [5] and therefore, the normal distribution formula was applied with $\psi = 0.018$ and $\sigma = 0.003$ as shown in Fig. 4.

$$F(t) = -Ae^{-(t-\psi)^2/\sigma} \tag{4}$$



(a) For t from 0 to 0.03



(b) For t from 0 to 0.003

Figure 4: Representation of the normal distribution force.

Matlab Code with numerical solutions for air-puff force model 1

Script 1:

```
function xdot=force1(t,x)
% state equations for numerical solution
m=15*10^-6;
k=85;
c=0.05;
a1=5149;
a2=3398;
b1=0.01752;
b2=0.001167;
c1=0.00441;
c2=0.04311;
A=7.068*10^-6;
FIOP = 13.8/7500*7.35;
xdot=zeros(2,1);
xdot(1)=x(2);
xdot(2)=-(k/m)*x(1)-(c/m)*x(2)-(a1*exp(-1*((t-b1)/c1).^2)
+a2*exp(-1*((t-b2)/c2).^2))*A/m +FIOP/m;
```

Command Window:

```
t0=0; tf=0.04;
x0=[13.8/7500*7.35/85 ; 13.8/7500*7.35/85];
tt = t0:0.0001:tf;
A = 7.068*10^-6;
Fair=(a1*exp (-1*((tt-b1)/c1).^2)
  +a2*exp(-1*((tt-b2)/c2).^2)).*A;
FIOP = 13.8/7500*7.35;
$[t,x]$=ode45('force1 ',[t0,tf],x0);
subplot(211),plot(tt,Fair,'-b');
grid on;
hold on;
plot (tt,FIOP,'--b');
xlabel('Time (sec)');
ylabel('Force (N)');
subplot(212),plot(t,x(:,1)*10^3,'-r');
grid on;
hold on;
xlabel('Time (sec)');
ylabel('Response (mm)');
legend('k=65 N/m','k=85 N/m','k=105 N/m','Location','SouthEast');
```

According to the Fig. 5 we can observe that for damping coefficient, c, equals to 0.05 Ns/m, and elasticity coefficient, k, equals to 65 N/m, the maximum force that was applied approximately at 0.0175 s is 0.058 N; regarding the force, the displacement of the cornea at its maximum of about 0.73 mm (it is in the negative direction because when the force is applied the cornea moves inward the eye).So, the external force remains constant and does not depends on elasticity and viscoelasticity coefficients. However, when the elasticity coefficient increases the cornea response decreases in displacement.
Fig. 6 compares the cornea response for c=0.01, 0.15, 0.26 Ns/m and k=65, 85, 105 N/m. When the viscoelasticity capacity increases to c=0.01 Ns/m, it starts to oscillate with an amplitude of 0.6 mm and it damps at 0.01 s; as k increases, the oscillation slightly reduces.

However, when the damping coefficient increases the curve from 0 to 0.01 s become smooth without any oscillations and goes down to 0.63 mm for k=65 N/m. So, as the elasticity coefficient grows up the amplitude of the displacement becomes smaller. For k=85 N/m

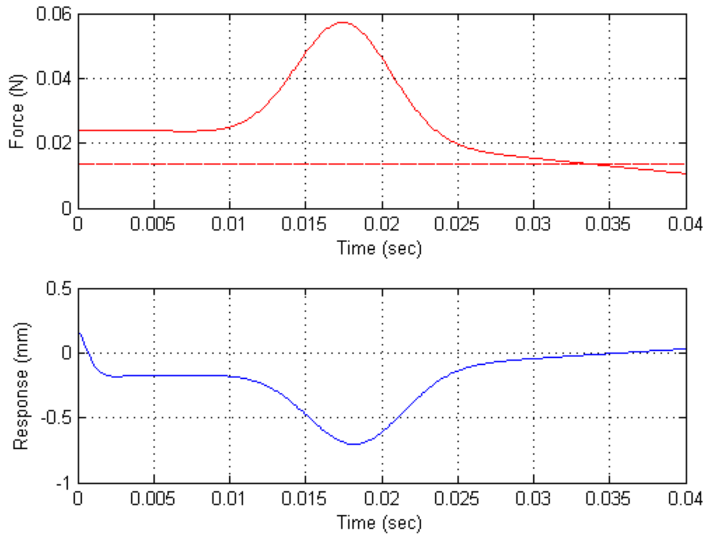
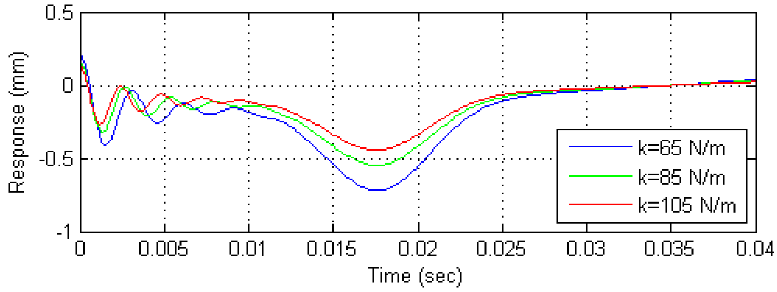
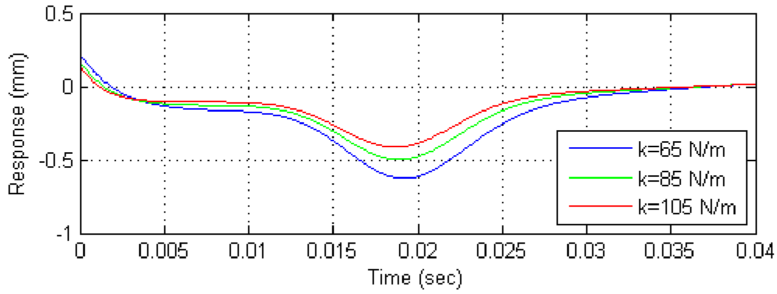


Figure 5: Total external force of IOP and response of the corneal movements at given time.

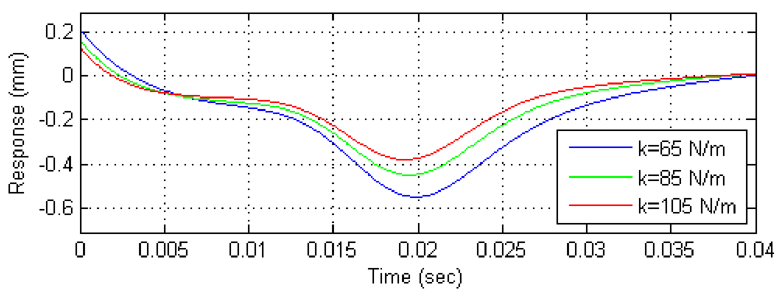
the displacement, y, equals to 0.50 mm and y=0.38 mm when k=105 N/m.
For c=0.26 Ns/m and k=65 N/m, the displacement slightly reduces to 0.57 mm and as it happens for other viscoelasticity coefficients, as k increases the displacement of the cornea also becomes smaller.
So, if we compare all three different cases of damping coefficient, it can be seen that as c increases the oscillation reduces and the curve becomes smoother that is shows the viscoelasticity characteristic. Moreover, for larger c the cornea changes less. We know that the ability to deform immediately shows a pure viscous behavior and the ability to return to its original place is a pure elastic behavior and so, when these two characteristics combined together they compensate each other and that is why an increase of the viscoelastic component causes to the change in the amplitude of the cornea displacement. The same happens when the elastic coefficient increases, but the displacement reduces more significantly. In addition, a viscoelastic material that is initially compliant becomes stiffer when the load is increased.



(a) c=0.01 Ns/m

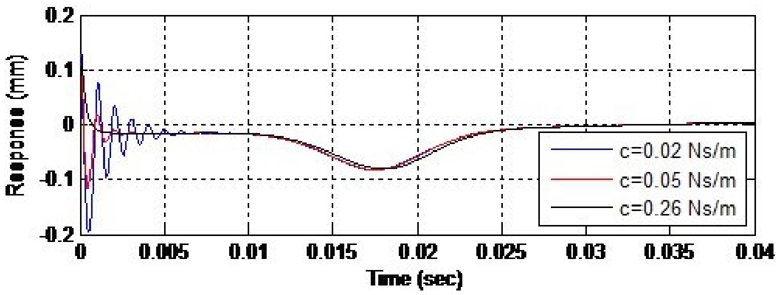


(b) c=0.05 Ns/m

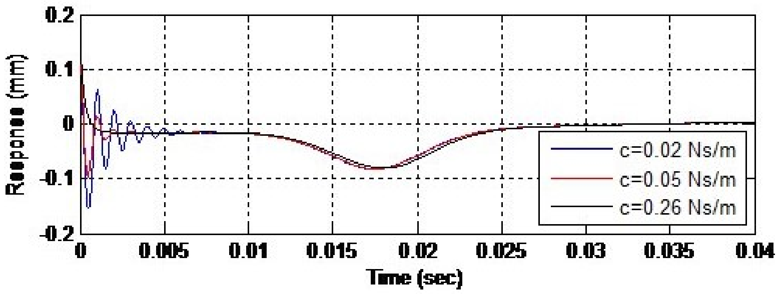


(c) c=0.26 Ns/m

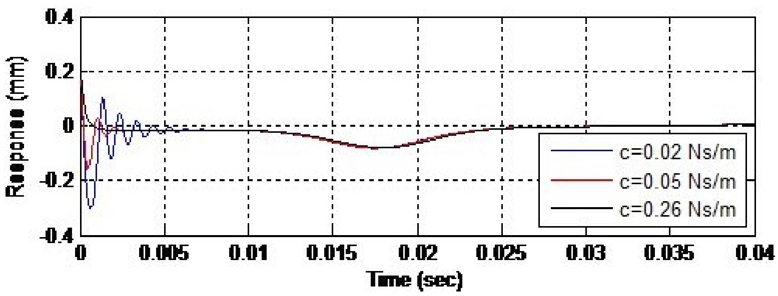
Figure 6: Time-dependent corneal movement response on force model 1 for different elasticity coefficients.



(a) k=65 N/m



(b) k=85 N/m



(c) k=105 N/m

Figure 7: Time-dependent corneal movement response on force model 1 for different viscoelasticity coefficients.

Matlab Code with numerical solutions for air-puff force model 2

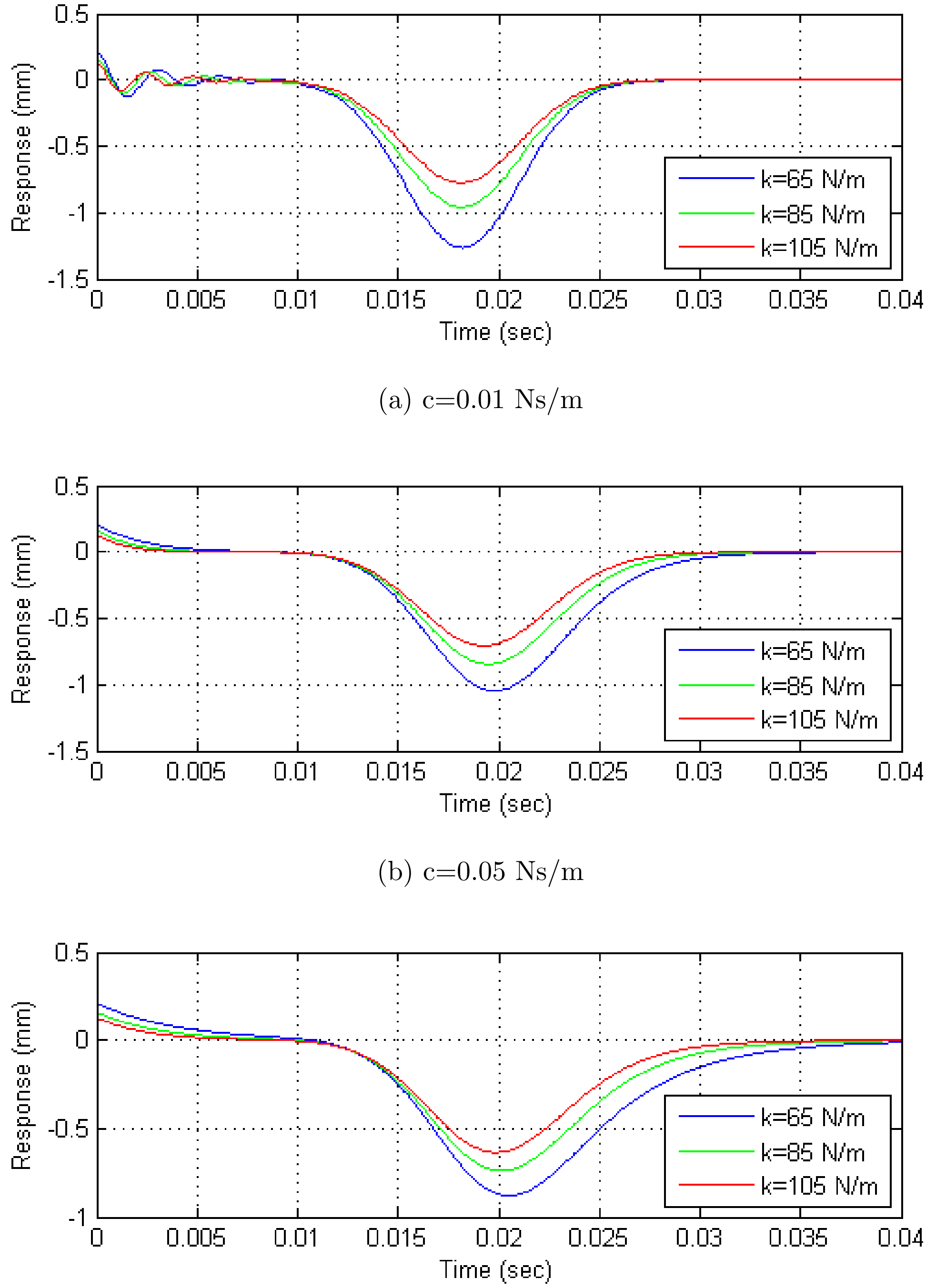
Script 2:

```
function xdot=force2(t,x)
m=15*10^-6;        %kg
k=105;             %N/m
c=0.26;            %Ns/m
xdot=zeros(2,1);
xdot(1)=x(2);
xdot(2)=-(k/m)*x(1)-(c/m)*x(2)- (0.08*exp(-((t-0.018)^2)/0.000018))/m;
```

Command Window:

```
t0=0; tf=0.04;
x0=[13.8/7500*7.35/105 ; 13.8/7500*7.35/105];
tt = t0:0.0001:tf;
[t,x]=ode45('force2',[t0,tf],x0);
subplot(212),plot(t,x(:,1)*10^3,'-r');
grid on;
hold on;
xlabel('Time (sec)');
ylabel('Response (mm)');
legend('k=65 N/m','k=85 N/m','k=105 N/m','Location','SouthEast');
```

When k (elasticity) increases the displacement decreases, it caused by the fact that the material tends to return to its original place faster. So, for c=0.01 Ns/m when k=65 N/m the amplitude of the response is about 1.25 mm and when k=105 N/m it becomes 0.75 mm.
Increase in viscoelasticity coefficient causes the more progressive deformation. If we compare the graphs for c=0.01, 0.05 and 0.26 Ns/m, we can notice that when the load is applied after reaching its peak, it requires more time to return to its initial position and the decreasing curve becomes more linear.

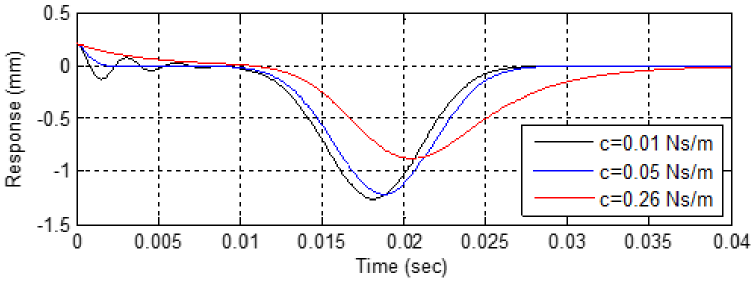


(a) c=0.01 Ns/m

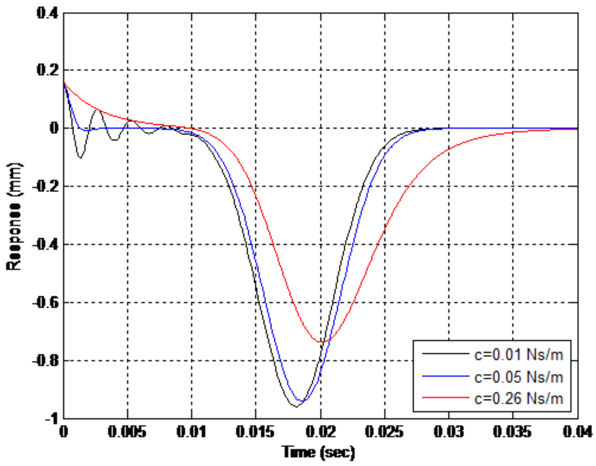
(b) c=0.05 Ns/m

(c) c=0.26 Ns/m

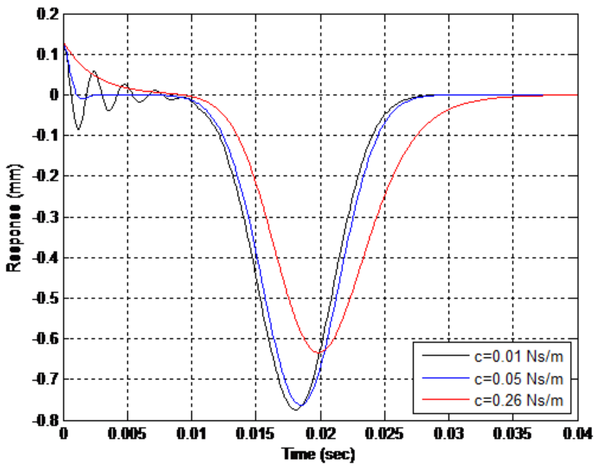
Figure 8: Time-dependent corneal movement response on force model 2 for different elasticity coefficients.



(a) k=65 N/m



(b) k=85 N/m



(c) k=105 N/m

Figure 9: Time-dependent corneal movement response on force model 2 for different viscoelasticity coefficients.

The Nonlinear Kelvin-Voight model

Given an exponential stress-strain relationship equation in order to determine how the strain rate affects on the material's stress-strain behaviour:

$$\sigma = \alpha(e^{\beta\varepsilon_1} - 1)$$

It is known that

$$f = A\sigma$$

and

$$\varepsilon = y/R$$

where f is a restoring force and $R = 7.8$ mm the radius from the center to the mid-cornea layer.
Using 2 terms of Taylor expansion we got the updated stress:

$$\sigma = \alpha(\frac{\beta\varepsilon_1}{R} + \frac{\beta^2\varepsilon_1^2}{2R^2}) \tag{5}$$

So, the updated dynamic model is

$$\begin{cases} m\frac{d^2y(t)}{dt^2} + c\frac{dy(t)}{dt} + \frac{A\alpha\beta y(t)}{R} = F(t) - A\frac{\alpha\beta^2}{2R^2}y^2 \\ y(0) = F_{IOP}/k \\ y(T) = F_{IOP}/k \end{cases} \tag{6}$$

k=85.275 N/m	α=0.0007425	β=38.5
k=20.337 N/m	0.8α=0.000594	0.8β=30.8
k=42.164 N/m	0.9α=0.00066825	0.9β=34.65
k=169.0326 N/m	1.1α=0.00081675	1.1β=42.35
k=329.542 N/m	1.2α=0.000891	1.2β=46.2

Figure 10: Calculated k and geometry constants (α,β).

Matlab Code with numerical solutions for air-puff force model 1

Script 3:

```
function xdot=non(t,x)
m=15*10^-6;       %kg
c=0.106;          %Ns/m
a1=5149;
a2=3398;
b1=0.01752;
b2=0.001167;
c1=0.00441;
c2=0.04311;
A=7.068*10^-6;
R=7.8*10^-3;
a=0.0007425;
b=38.5;
FIOP = 13.8/7500*7.35;
xdot=zeros(2,1);
xdot(1)=x(2);
xdot(2)=-(A*a*b/(R*m))*x(1)-(c/m)*x(2)- (a1*exp
(-1*((t-b1)/c1).^2)+a2*exp(-1*((t-b2)/c2).^2))*A/m
+FIOP/m-(A*a*b.^2/(2*R.^2*m))*(x(1).^2);
```

Command Window:

```
t0=0; tf=0.04;
x0=[13.8/7500*7.35/85.27501368; 13.8/7500*7.35/85.27501368];
tt = t0:0.0001:tf;
Fair=(a1*exp (-1*((tt-b1)/c1).^2)+a2*exp(-1*((tt-b2)/c2).^2)).*A;
FIOP = 13.8/7500*7.35;
[t1, y]=ode45('Febs',[t0,tf],x0);
[t,x]=ode45('np1',[t0,tf],[x0; y(1)]);
subplot(211),plot(tt,Fair,'-r');
grid on;
hold on;
plot (tt,FIOP,'--r');
xlabel('Time (sec)'); ylabel('Force (N)');
```

```
subplot(212),plot(t,x(:,1)*10^3,'-y');
grid on;
hold on;
xlabel('Time (sec)'); ylabel('Response (mm)');
legend('0.8a; 0.8b','0.9a; 0.9b','a; b','1.1a; 1.1b','1.2a; 1.2b',
'Location','SouthEast');
```

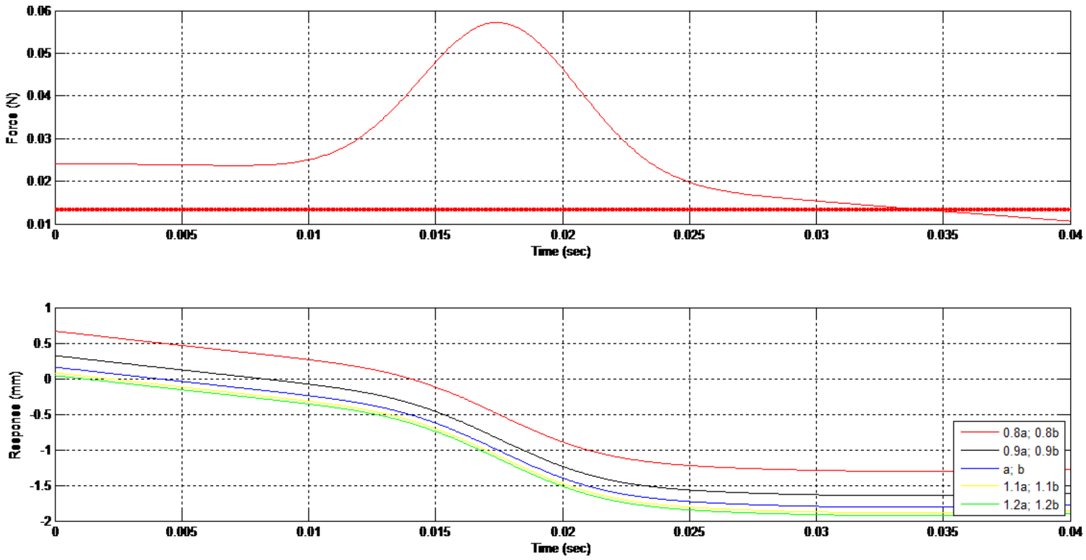


Figure 11: Time-dependent corneal movement response on force model 1 for different geometric constants.

Matlab Code with numerical solutions for air-puff force model 2

Script 4:

```
function xdot=force2(t,x)
m=15*10^-6;        %kg
k=329.5420916;     %N/m
c=0.106;           %Ns/m
A=7.068*10^-6;
R=7.8*10^-3;
a=0.000891;
```

```
b=46.20;
xdot=zeros(2,1);
xdot(1)=x(2);
xdot(2)=-(A*a*b/(R*m))*x(1)-(c/m)*x(2)-
(0.08*exp(-((t-0.018)^2)/0.000018))/m-
(A*a*b.^2/(2*R.^2*m))*(x(1).^2);
```

Command Window:

```
t0=0; tf=0.04;
x0=[13.8/7500*7.35/329.5420916; 13.8/7500*7.35/329.5420916];
tt = t0:0.0001:tf;
[t,x]=ode45('force2',[t0,tf],x0);
plot(t,x(:,1)*10^3,'-g');
grid on;
hold on;
xlabel('Time (sec)');
ylabel('Response (mm)');
legend('0.8a; 0.8b','0.9a; 0.9b','a; b','1.1a; 1.1b',
'1.2a; 1.2b','Location','SouthEast');
```

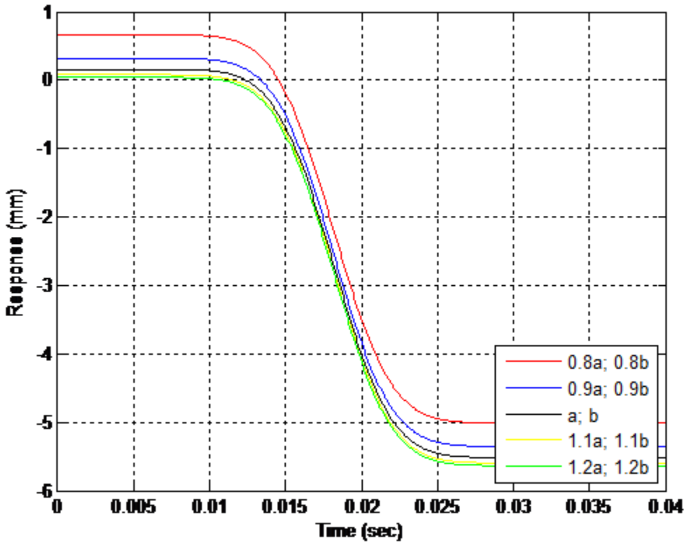


Figure 12: Time-dependent corneal movement responce on force model 2 for different geometric constants.

We can give a k value of 85.27 N/m with $\alpha = 7.425 * 10^{-4}$ and $\beta = 38.50$ [6]. So now, we can vary α and β with a factor of 0.8 to 1.2 (e.g. 0.8α to 1.2 and 0.8β and 1.2β) and discuss the contribution of α and β to the overall corneal deformation behavior under the same damping coefficient. Damping coefficient is the mean value from the linear part, c=(0.01+0.05+0.26)/3=0.106 Ns/m. As we can see from the table, our k positively dependent from alpha and beta, where $k = \frac{E*t^2}{a*(R-(t/2))*(1-v)^{1/2}}$ [7]. According to the Fig. 11 after the applying force displacement ranges approximately between 0.7 and 0.05 mm at the beginning. The graphs show the steady decline before t=0.015 s, after gradually decline for 0.07 s. From 0.025 s to 0.04 s graphs stay stable at 1.8 mm displacement. When the geometry constants, α and β, increases the cornea response increases in displacement. It is reasonable, because our k also increases.

FUTURE LINES OF WORK IN THIS AREA: Constructing Nonlinear Maxwell model

The second model that will be considered for constructing human eye ball is a nonlinear Maxwell model composing a nonlinear spring connected in series with a nonlinear dashpot keeping a power-law with constant material parameters. By using this model we represent successful time-dependent properties of a diversity of viscoelastic materials. Woo and Kobayashi [6] determined the isotropy of the corneal material by measuring the lateral deformations. It was found that in all three directions the deformation is very small. So to obtain in a three dimensional state the stress-strain curves, a mathematical model, using the the effective stress and effective trilinear deformation, was constructed.

The formulae for effective stress and effective strain [6] are

$$\sigma_e = \frac{1}{3}\sqrt{(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2}$$
$$\varepsilon_e = \frac{2}{3}\sqrt{(\varepsilon_1-\varepsilon_2)^2+(\varepsilon_2-\varepsilon_3)^2+(\varepsilon_3-\varepsilon_1)^2}$$

and the following exponential function was obtained

$$\sigma = p(e^{q\varepsilon_1} - 1). \tag{7}$$

In uniaxial case

$$\sigma_e = \sigma_1,$$
$$\varepsilon_e = \varepsilon_1.$$

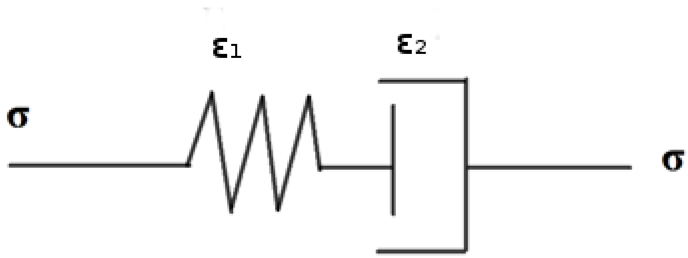


Figure 13: The proposed rheological model [9].

In the following calculations we used σ for σ_1.

From [8] given that σ is a total stress, where $\sigma = E\varepsilon_1^m$, and ε is a total strain. So instead of the given stress we substituted the Eq. (7) and adapted the following model [8]:

$$\begin{cases} \sigma = p(e^{q\varepsilon_1} - 1) \\ \sigma = \eta(\varepsilon_2^{'})^{\frac{1}{n}} \\ \varepsilon = \varepsilon_1 + \varepsilon_2 \end{cases} \tag{8}$$

where ε_1 and ε_2 are the strains of the nonlinear spring and nonlinear dashpot; η is the viscosity module; m and n are nonlinearity parameters; α and β are constants with values $5.4 * 10^{-4}$ and 41.8, respectively [6].

Then ε_1 and ε_2 are found, a differentiation with respect to time is performed and then we substituted it in the Eq. (8):

$$\begin{aligned} \sigma &= p(e^{q\varepsilon_1} - 1) \\ \frac{\sigma}{p} + 1 &= e^{q\varepsilon_1} \\ \log(\frac{\sigma}{p} + 1) &= q\varepsilon_1 \\ \varepsilon_1 = \frac{1}{q}\log(\frac{\sigma}{p} + 1) &= \log(\frac{\sigma + p}{p})^{\frac{1}{q}} \end{aligned}$$

$$u = \frac{\sigma + p}{p}$$
$$du = \frac{1}{p} d\sigma$$
$$\varepsilon_1^{'} = (\frac{1}{q}\log(u))^{'} = \frac{\sigma^{'}}{q(\sigma + p)}$$
$$\varepsilon^{'} = \varepsilon_1^{'} + \varepsilon_2^{'}$$
$$\varepsilon^{'} = \frac{y'}{R}$$

$$\varepsilon^{'} = \frac{1}{q(\sigma + p)}\sigma^{'} + |\frac{\sigma}{\eta}|^{n-1}(\frac{\sigma}{\eta}) = \frac{y'}{R} \tag{9}$$

So, our first order differential equation that represents the relation between the total stress σ and the strain ε of the material is:

$$\sigma^{'} + q(\sigma + p)|\frac{\sigma}{\eta}|^{n-1}(\frac{\sigma}{\eta}) = \frac{y'}{R}q(\sigma + p) \tag{10}$$

In the next paper we will solve this nonlinear Maxwell model and will attempt to find the relationship between stress and strain.

Conclusion

To sum up, we duplicated the corneal movements that were simulated under the air puff tonometry. For this purpose, two external forces, which are originally was taken from Han's and Nishiyama's papers for comparative purposes, was applied to calculate the kinematic non-homogeneous second order differential equation. All parametric and experimental results also was provided. However, the model was modified by Taylor expansion up to quadratic terms and was solved by using the MatLab and Maple. It was found that the larger values of viscoelasticity, c, and elasticity, k, coefficients are both affect on the performance of corneal movements: increase in c causes the damping in the oscillation and as k increases the displacement of the cornea decreases. Moreover, we attempted to solve the nonlinear model by using the exponential stress-strain function that will be studied more deeply in the next paper.

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Bibliography

[1] Kingman, S. (2004). Glaucoma is second leading cause of blindness globally, Bulletin of the World Health Organization, 82, 887-888.

[2] Somkiat, A. (2012). Glaucoma. Retrieved April 07, 2016, from http://glaucoma.doctorsomkiat.com/englo3.htm

[3] Girard, M. J., Downs, J. C., Bottlang, M., Burgoyne, C. F. and Suh, J. F. (2009). Peripapillary and Posterior Scleral MechanicsPart II: Experimental and Inverse Finite Element Characterization. Journal of Biomechanical Engineering J. Biomech. Eng., 131(5), 051012.

[4] Han, Z., Tao C., Zhou, D., Sun, Y., Zhou, C., Ren, Q. and Roberts, C. (2014). Air Puff Induced Corneal Vibrations: Theoretical Simulations and Clinical Observations. Journal Of Refractive Surgery, 30(3), 208-213.

[5] Nishiyama, J., Kao, I., and Kaneko, M. (2013). IOP measurement using air-puff tonometry: Dynamic modeling of human eyeball with experimental results. 2013 10th International Conference on Ubiquitous Robots and Ambient Intelligence (URAI).http://ieeexplore.ieee.org/stamp/stamp.jsp?arnumber=6677496tag=1

[6] Woo,S.L-Y.,Kobayashi, A. S., Schlegel, W. A. and Lawrence, C.(1972). Nonlinear Material Properties of Intact Cornea and Sclera,Exp. Eye Res., 14, 29-39.

[7] Match, W. L. Ko., Leo, K. K., David, C. C., Christopher, K.S.(2013). Characterization of corneal tangent modulus in vivo,Acta Ophthalmologica, 263-269.

[8] Monsia., M. D. (2011). A Simplified Nonlinear Generalized Maxwell Model for Predicting the Time Dependent Behavior of Viscoelastic Materials. World Journal of Mechanics, 1(1), 158-167.